Proceedings of the Jangjeon Mathematical Society
20 (2017), No. 2. pp. 213 – 225

www.jangjeon.or.kr
http://dx.doi.org/10.23001/pjms2017.20.2.213

STRONG CONVERGENCE THEOREMS OF ITERATIVE ALGORITHMS FOR RELATIVELY NON-LIPSCHITZIAN MAPPINGS

HYE JEONG CHA AND TAE-HWA KIM*

Abstract. In this paper, we first review strong convergence theorems of iterative algorithms due to Matsushita and Takahashi [20] for relatively nonexpansive mappings and next extend their convergence results to the wider class of uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive type. Finally we discuss some applications relating to our main result.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. Then T is said to be a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$. A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* [9] if $\lim_{n\to\infty} k_n = 1$, respectively.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping of *asymptotically nonexpansive type* [15] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.1}$$

and T^N is continuous for some $N \geq 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [3]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.2}$$

Remark 1.1. In the case of (1.1), for each $x \in C$, if we define

$$c_n(x) := \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

2010 *Mathematics Subject Classification.* 46B20, 46N10, 47H09.

Key words and phrases. Iterative algorithm, mappings of asymptotically nonexpansive type, relatively nonexpansive mappings, and strong convergence.

*Corresponding author. Supported by a Research Grant of Pukyong National University (2016 year).

where $a \vee b := \max\{a, b\}$, then $c_n(x) \geq 0$ for all $n \geq 1$, $c_n(x) \to 0$ as $n \to \infty$ for each $x \in C$, and thus (1.1) immediately reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n(x) \tag{1.3}$$

for all $y \in C$ and $n \geq 1$. Observe that the converse always remains true, namely, (1.3) also implies (1.1). Indeed, (1.3) implies

$$\|T^n x - T^n y\| - \|x - y\| \leq c_n(x), \quad y \in C,$$

i.e., $c_n(x)$ is an upper bound of $\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}$ and thus

$$\sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|) \leq c_n(x),$$

which implies (1.1) since $c_n(x) \to 0$. Similarly, if we define

$$c_n := \sup_{x \in C} c_n(x)$$

for each $n \geq 1$, note that (1.2) is equivalent to the following (1.4)

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$, where $\{c_n\}$ is a sequence of nonnegative real numbers such that $c_n \to 0$ as $n \to \infty$.

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T [24] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Let X be a smooth Banach space and let X^* be the dual of X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$, where J is the normalized duality mapping from X to X^*. Recall that a mapping $T : C \to C$ is *relatively asymptotically nonexpansive* (denoted simply by RAN) [14] if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and, for each $n \geq 1$ there exists a constant $k_n > 0$ such that

$$\phi(p, T^n x) \leq k_n^2 \phi(p, x) \tag{1.5}$$

for $x \in C$ and $p \in F(T)$, where $\lim_{n\to\infty} k_n = 1$. In particular, T is called *relatively nonexpansive* [19] if $k_n = 1$ for all n; see also [3,4,5].

Motivated and initiated by (1.3) and (1.5), we say that $T : C \to C$ is a mapping of *relatively asymptotically nonexpansive type* (denoted simply by $RANT$) if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and, for each $x \in C$ there exists a sequence $\{c_n(x)\}$ of nonnegative real numbers, $c_n(x) \to 0$, such that

$$\phi(p, T^n x) \leq \phi(p, x) + c_n(x) \tag{1.6}$$

for all $p \in F(T)$ and $n \geq 1$.

Remark 1.2. Observe that if $T: C \to C$ is RAN and $F(T)$ is bounded, then it is clearly a mapping of $RANT$ by taking

$$c_n(x) = (k_n^2 - 1) \sup_{p \in F(T)} \phi(p, x) \to 0$$

as $n \to \infty$ for each $x \in C$.

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [10] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \tag{1.7}$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [17] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.8}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \tag{1.9}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. By taking $\beta_n = 1$ for all $n \geq 0$ in (1.9), Ishikawa's iteration process reduces to the Mann's iteration process (1.8). It is known in [6] that the process (1.8) may fail to converge while the process (1.9) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (1.7) has been proved to be strongly convergent in both Hilbert spaces [10, 16, 28] and uniformly smooth Banach spaces [22, 25, 30], while Mann's iteration (1.8) has only weak convergence even in a Hilbert space [8].

Attempts to modify the Mann iteration method (1.8) or the Ishikawa iteration method (1.9) so that strong convergence is guaranteed have been made. In 2003, Nakajo and Takahashi [21] proposed the following modification of Mann's iteration process (1.8) for a single nonexpansive mapping T

with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\left\{\begin{array}{l} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{array}\right. \tag{1.10}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{F(T)}x_0$. As a special case, taking $\alpha_n = 0$ for all n, the above iteration scheme (1.10) reduces to the following:

$$\left\{\begin{array}{l} x_0 \in C \text{ chosen arbitrarily}, \\ C_n = \{z \in C : \|Tx_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{array}\right. \tag{1.11}$$

On the other hand, Kamimura and Takahashi [12] considered the problem of finding an element u of a Banach space X satisfying $0 \in Au$, where $A \subset X \times X^*$ is a maximal monotone operator and X^* is the dual space of X. They studied the following algorithm:

$$\left\{\begin{array}{l} x_0 \in X \text{ chosen arbitrarily}, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n),\ v_n \in Ay_n, \\ H_n = \{z \in X : \langle y_n - z, v_\rangle \geq 0\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{array}\right. \tag{1.12}$$

where J is the duality mapping on X, $\{r_n\}$ is a sequence of positive real numbers and Π_K denotes the generalized projection from X onto a nonempty closed convex subset K of X; see the section 2 for more details. They proved that if $A^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to an element of $A^{-1}0$. This generalizes the result due to Solodov and Svaiter [26] in a Hilbert space.

In 2005, Matsushita and Takahashi [19] extended Nakajo and Takahashi's iteration process (1.10) to the following modification of Mann's iteration process (1.8) using the hybrid method in mathematical programming for a relatively nonexpansive mapping $T : C \to C$ in a uniformly convex and uniformly smooth Banach space X:

$$\left\{\begin{array}{l} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{array}\right. \tag{1.13}$$

where J is the normalized duality mapping. Then they proved that if the sequence $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to $\Pi_{F(T)} x_0$, where Π_K denotes the generalized projection from X onto a closed convex subset K of X. As a special case, taking $\alpha_n = 0$ for all n in (1.13), the iteration scheme reduces to the following:

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
H_n = \{z \in C : \phi(z, Tx_n) \le \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x_0,
\end{cases}
\tag{1.14}
$$

which generalizes the iteration scheme (1.11) in a Hilbert spaces. Very recently, they also established that even though the condition of uniformly smooth of X is only weakened by the smooth condition of X, the sequence $\{x_n\}$ generated by (1.14) still converges strongly to $\Pi_{F(T)} x_0$.

The purpose of this paper is to employ the idea due to Matsushita and Takahashi [20] to prove some strong convergence theorems for uniformly Lipschitzian mappings which are RANT in uniformly convex and smooth Banach spaces. The paper is organized as follows. In the next section we introduce some lemmas and propositions studied recently in [12] and [13, 14] which play crucial roles for our argument. In Section 3, motivated by [20], we extend strong convergence theorems of iterative algorithm (1.14) due to Matsushita and Takahashi to those for the wider class of uniformly Lipschitzian mappings which are RANT and discuss some applications relating to our main result.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle \cdot, \cdot \rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$. The normalized duality mapping J from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

A Banach space X is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all x, $y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\, y \in U$. It is also known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X. Some properties of the duality mapping have been given in [7, 23, 27]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [7, 27] for more details.

Let X be a smooth Banach space. Recall that the function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \tag{2.2}$$

for all $x, y \in X$. Further, we have that for any $x, y, z \in X$,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, J(z) - J(y)\rangle.$$

In particular, it is easy to see that if X is strictly convex, for $x, y \in X$, $\phi(y, x) = 0$ if and only if $y = x$ (see, for example, Remark 2.1 of [19]).

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of X. Then, for any $x \in X$, there exists a unique element $\tilde{x} \in C$ such that

$$\phi(\tilde{x}, x) = \inf_{z \in C} \phi(z, x).$$

Then a mapping $\Pi_C : X \to C$ defined by $\Pi_C x = \tilde{x}$ is called the *generalized projection* (see [1, 2, 12]). In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1, 2, 12]).

Proposition 2.1. ([1, 2, 12]) *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If X is smooth, then, $\tilde{x} = \Pi_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \ge 0$ for $y \in K$.*

(b) *If X is reflexive, strictly convex and smooth, then*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \le \phi(y, x)$$

for all $y \in K$.

The following subsequent two lemmas are motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [18] in Hilbert spaces, respectively; for detailed proofs, see [13].

Lemma 2.2. ([13]) *Let C be a nonempty closed convex subset of a smooth Banach space X, $x, y, z \in X$ and $\lambda \in [0, 1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \le \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

is closed and convex.

Lemma 2.3. ([13]) *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X. Let $x_0 \in X$ and $q := \Pi_K x_0$, where Π_K denotes the generalized projection from X onto K. If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0)$$

for all n. Then $x_n \to q \, (= \Pi_K x_0)$.

In 2003, Kamimura and Takahashi [12] proved the following result, which plays a crucial role in our discussion.

Proposition 2.4. ([12]) *Let X be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of X. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Finally, concerning the set of fixed points of a mapping of RANT, we shall prove the following result.

Proposition 2.5. *Let X be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a continuous mapping of RANT. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence of $F(T)$ such that $x_n \to x \ \in C$. Since T is a mapping of RANT, we have that

$$\phi(x_n, T^m x) \quad \leq \quad \phi(x_n, x) + c_m(x)$$

for each $n, m \geq 1$. Fix $m \geq 1$. Firstly taking the limit on both sides as $n \to \infty$, we have

$$\begin{aligned} \phi(x, T^m x) &= \lim_{n\to\infty} \phi(x_n, T^m x) \leq \lim_{n\to\infty} [\phi(x_n, x) + c_m(x)] \\ &= \phi(x, x) + c_m(x) = c_m(x). \end{aligned}$$

As taking the limit on both sides as $m \to \infty$, since $c_m(x) \to 0$ as $m \to \infty$. It follows from Proposition 2.4 that $T^m z \to z$ as $m \to \infty$ and hence $z \in F(T)$ by the continuity of T. Next, we show that $F(T)$ is convex. For $x, y \in F(T)$ and $\lambda \in (0, 1)$, put $z = \lambda x + (1 - \lambda)y$. It suffices to show that $z \in F(T)$. Indeed, as in [19], we have that for $n \geq 1$,

$$\begin{aligned} &\phi(z, T^n z) = \|z\|^2 - 2\langle z, JT^n z\rangle + \|T^n z\|^2 \\ = \ &\|z\|^2 - 2\langle \lambda x + (1-\lambda)y, JT^n z\rangle + \|T^n z\|^2 \\ = \ &\|z\|^2 - 2\lambda\langle x, JT^n z\rangle - 2(1-\lambda)\langle y, JT^n z\rangle + \|T^n z\|^2 \\ = \ &\|z\|^2 + \lambda\phi(x, T^n z) + (1-\lambda)\phi(y, T^n z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 \\ \leq \ &\|z\|^2 + [\lambda\phi(x, z) + (1-\lambda)\phi(y, z)] + 2c_n(z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2. \end{aligned}$$

Since $c_n(z) \to 0$, the right hand side of the above inequality converges to 0 because

$$\begin{aligned} & \|z\|^2 + \lambda\phi(x,z) + (1-\lambda)\phi(y,z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2. \\ = \; & \|z\|^2 - 2\langle \lambda x + (1-\lambda)y, Jz\rangle + \|z\|^2 \\ = \; & \|z\|^2 - 2\langle z, Jz\rangle + \|z\|^2 = 0 \end{aligned}$$

By Proposition 2.4 again, we have $T^n z \to z$ and hence $z \in F(T)$ by the continuity of T. □

3. Strong convergence theorems

In this section we first propose an iteration process, motivated by the idea due to [20], to have strong convergence for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive in the intermediate sense in uniformly convex and smooth Banach spaces.

Theorem 3.1. *Let X be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping of RANT. Assume that $F(T) \neq \emptyset$. Let a sequence $\{x_n\}$ in C be defined by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \{v \in C : \phi(v, T^n x_n) \leq \phi(v, x_n) + c_n(x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0. \end{cases}$$

Then the sequence $\{x_n\}$ converges in norm to $\Pi_{F(T)} x_0$.

Proof. First, observe that H_n is closed and convex by Lemma 2.2, and that W_n is obviously closed and convex for each $n \geq 0$. Next we show that $F(T) \subset H_n$ for all n. Indeed, for all $p \in F(T)$, Since T is a mapping of RANT, we get

$$\phi(p, T^n x_n) \quad \leq \quad \phi(p, x_n) + c_n(x_n)$$

and so $p \in H_n$; hence $F(T) \subset H_n$ for all $n \geq 0$. Moreover, we show that

$$F(T) \subset H_n \cap W_n \tag{3.1}$$

for all $n \geq 0$. It suffices to show that $F(T) \subset W_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F(T) \subset C = W_0$. Assume that $F(T) \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the generalized projection of x_0 onto $H_k \cap W_k$, by Proposition 2.1 (a) we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1}\rangle \geq 0$$

for all $z \in H_k \cap W_k$. As $F(T) \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of W_{k+1} implies that $F(T) \subset W_{k+1}$. Hence (3.1) holds for all $n \geq 0$. So, $\{x_n\}$ is well defined. Obviously, since $x_n = \Pi_{W_n} x_0$ by the definition of W_n and

Proposition 2.1 (a), and $F(T) \subset W_n$, we have $\phi(x_n, x_0) \leq \phi(p, x_0)$ for all $p \in F(T)$. In particular, we obtain, for all $n \geq 0$,

$$\phi(x_n, x_0) \leq \phi(q, x_0), \qquad \text{where } q := \Pi_{F(T)} x_0. \tag{3.2}$$

Therefore, $\{\phi(x_n, x_0)\}$ is bounded; so is $\{x_n\}$ by (2.2).

Noticing that $x_n = \Pi_{W_n} x_0$ again and the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$, we get

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0),$$

which shows that the sequence $\{\phi(x_n, x_0)\}$ is increasing and so the $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Simultaneously, from Proposition 2.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi\left(x_{n+1}, \Pi_{W_n} x_0\right) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0. \end{aligned} \tag{3.3}$$

By Proposition 2.4, we have

$$\|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

Now since $x_{n+1} \in H_n$, it follows from the definition of H_n, (3.3) and $c_n(x_n) \to 0$ by virtue of Cantor's diagonal process that

$$\phi(x_{n+1}, T^n x_n) \leq \phi(x_{n+1}, x_n) + c_n(x_n) \to 0.$$

Using Proposition 2.4 again yields

$$\|x_{n+1} - T^n x_n\| \to 0$$

and this combined with (3.4) gives

$$\|x_n - T^n x_n\| \to 0. \tag{3.5}$$

Since T is uniformly k-Lipschitzian, it follows from (3.4) and (3.5) that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\ &\leq (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + k\|T^n x_n - x_n\| \to 0. \end{aligned} \tag{3.6}$$

By (3.6), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. This, combined with (3.2) and Lemma 2.3 (with $K = F(T)$), guarantees that $x_n \to q = \Pi_{F(T)} x_0$. The proof is complete. □

As a direct consequence of Remark 1.2 and Theorem 3.1 we have the following

Corollary 3.2. *Let X be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly*

k-Lipschitzian mapping of RAN. Assume that $F(T)$ is a nonempty bounded subset of C. Let a sequence $\{x_n\}$ in C be defined by the following algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \{v \in C : \phi(v, T^n x_n) \le \phi(v, x_n) + c_n(x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases}$$

where $c_n(x_n) = (k_n^2 - 1) \cdot \sup\{\phi(p, x_n) : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges in norm to $\Pi_{F(T)} x_0$.

Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then, after noticing that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$, we see that $\|T^n x - T^n y\| \le k_n \|x - y\|$ is equivalent to $\phi(T^n x, T^n y) \le k_n^2 \phi(x, y)$. It is therefore easy to show that every asymptotically nonexpansive mapping is both uniformly k-Lipschitzian and RAN. In fact, it suffices to show that $\hat{F}(T) \subset F(T)$. The inclusion follows easily from the well-known demiclosedness at zero of $I - T$ (c.f., [29]), where I denotes the identity operator. Thus we have the following Hilbert space's version of Corollary 3.2.

Corollary 3.3. *Let X be a Hilbert space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $F(T)$ is a nonempty bounded subset of C. Let a sequence $\{x_n\}$ in C be defined by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_n = \{v \in C : \|v - T^n x_n\|^2 \le \|v - x_n\|^2 + \eta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\eta_n = (k_n^2 - 1) \cdot \sup\{\|p - x_n\|^2 : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges in norm to $P_{F(T)} x_0$, where $P_{F(T)}$ is the metric projection from X onto $F(T)$.

Finally, as a slight modification of Theorem 3.1, we propose another iteration process to have strong convergence for uniformly Lipschitzian mappings of RANT in uniformly convex and smooth Banach spaces.

Theorem 3.4. *Let X be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping of RANT. Assume that $F(T)$ is nonempty. Let a sequence $\{x_n\}$ in C be defined by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ H_n = \{v \in C : \limsup_{i \to \infty} [\phi(v, T^i x_n) - \phi(v, x_n)] \le 0\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $\Pi_{F(T)} x_0$.

Proof. We first show that H_n is closed and convex. In fact, the closedness of H_n is obvious from the continuity of $\phi(\cdot, x)$ for $x \in X$. Let us show that H_n is convex. As a matter of fact, the defining inequality in H_n is equivalent to the inequality

$$\limsup_{i\to\infty}[2\langle v, Jx_n - JT^i x_n\rangle + \|T^i x_n\|^2 - \|x_n\|^2] \le 0.$$

Thus, H_n is clearly convex.

Next we show that $F(T) \subset H_n$ for all n. Indeed, for all $p \in F(T)$, Since T is a mapping of RANT,

$$\phi(p, T^i x_n) - \phi(p, x_n) \le c_i(x_n)$$

and taking the lim sup on the both sides as $i \to \infty$, the right side converges to 0 for each $n \ge 1$ and so $p \in H_n$; hence $F(T) \subset H_n$ for all $n \ge 0$. Moreover, using the same processes of the proof of Theorem 3.1, we can show that

$$F(T) \subset H_n \cap W_n \tag{3.7}$$

for all $n \ge 0$, and furthermore (3.2)-(3.4). Now since $x_{n+1} \in H_n$, from the definition of H_n, we have

$$\limsup_{i\to\infty}[\phi(x_{n+1}, T^i x_n) - \phi(x_{n+1}, x_n)] \le 0,$$

and so

$$\lim_{n\to\infty}\limsup_{i\to\infty}\phi(x_{n+1}, T^i x_n) = 0.$$

Then it is not hard to see that there exists a $j \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{n\to\infty}\phi(x_{n+1}, T^{n+j} x_n) = 0.$$

Using Proposition 2.4 again yielding

$$\|x_{n+1} - T^{n+j} x_n\| \to 0$$

and this combined with (3.5) gives

$$\|x_n - T^{n+j} x_n\| \to 0. \tag{3.8}$$

Since T is uniformly k-Lipschitzian, it follows from (3.4) and (3.8) that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+j+1}x_{n+1}\| \\ &\quad + \|T^{n+j+1}x_{n+1} - T^{n+j+1}x_n\| + \|T^{n+j+1}x_n - Tx_n\| \\ &\le (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1+j}x_{n+1}\| \\ &\quad + k\|T^{n+j}x_n - x_n\| \to 0. \end{aligned} \tag{3.9}$$

By (3.9), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. This, combined with (3.2) and Lemma 2.3 (with $K = F(T)$), guarantees that $x_n \to q = \Pi_{F(T)}x_0$. The proof is complete. □

References

[1] Ya. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15–50.

[2] Ya. I. Alber and S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, *PanAmer. Math. J.,* **4** (1994), 39–54.

[3] R. E. Bruck, T. Kuczumow and S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, *Colloq. Math.,* **65** (1993), 169–179.

[4] D. Butnariu, S. Reich and A. J. Zaslavski, Asymptotic behavior of relatively nonexpansive operators in Banach spaces, *J. Appl. Anal.,* **7** (2001), 151–174.

[5] D. Butnariu, S. Reich and A. J. Zaslavski, Weak convergence of orbits of nonlinear operators in reflexive Banach spaces, *Numer. Funct. Anal. Optim.,* **24** (2003), 489–508.

[6] C. E. Chidume and S. A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, *Proc. Amer. Math. Soc.,* **129** (2001), 2359-2363.

[7] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990.

[8] A. Genel and J. Lindenstrass, An example concerning fixed points, *Israel J. Math.,* **22** (1975), 81-86.

[9] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **35** (1972), 171-174.

[10] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.,* **73** (1967), 957–961.

[11] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.,* **44** (1974), 147-150.

[12] S. Kamimura and W. Takahashi, Strong convergence of a proxiaml-type algorithm in a Banach space, *SIAM J. Optim.,* **13** (2003), 938–945.

[13] T. H. Kim and H. J. Lee, Strong Convergence of modified iteration processes for relatively nonexpansive mappings, *Kyungpook Math. J.*, **48** (2008), 685–703.

[14] T. H. Kim and W. Takahashi, Strong convergence of modified iteration processes for relatively asymptotically nonexpansive mappings, *Taiwanese J. Math.,* **14** (2010), 2163-2180.

[15] W. A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, *Israel J. Math.,* **17** (1974), 339–346.

[16] P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Sèr. A-B Paris* **284** (1977), 1357–1359.

[17] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.,* **4** (1953), 506-510.

[18] C. Martinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.*, **64** (2006), 2400–2411.

[19] S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory,* **134** (2005), 257–266.

[20] S. Matsushita and W. Takahashi, An iterative algorithm for relatively nonexpansive mappings by a hybrid method and applications, in: W. Takahashi and T. Tanaka (Ed.), *Nonlinear Analysis and Convex Analysis, Tokyo 2003*, Yokohama Publ., Yokohama, 2004, pp. 305–313.

[21] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.,* **279** (2003), 372-379.

[22] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.,* **75** (1980), 287–292.

[23] S. Reich, Review of Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990, *Bull. Amer. Math. Soc.,* **26** (1992), 367–370.

[24] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, in:A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313–318.

[25] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.,* **125** (1997), 3641–3645.

[26] M. V. Solodov and B. F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Program.,* **87** (2000), 189–202.

[27] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[28] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.,* **58** (1992), 486–491.

[29] H. K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, *Nonlinear Anal.,* **16** (1991), 1139–1146.

[30] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.,* **66** (2002), 240-256.

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea
E-mail address: pingk54@naver.com

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea
E-mail address: taehwa@pknu.ac.kr